Solving the dynamic user equilibrium problem via sequential convex optimization for parallel horizontal queuing networks

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ABSTRACT

This article considers the dynamic user equilibrium (DUE) problem for parallel networks. The network dynamics are modeled using a Godunov discretization of the Lighthill-Williams-Richards partial differential equation with a trapezoidal flux function. The model is augmented with an additional constraint that prevents vehicle holding which is a flaw in the discretization. The departure rates are assumed to be fixed. Under these assumptions, we show that the future allocation of the demand among the different paths at the origin has no effect on the travel time of the vehicles already in the network. This enables us to show that the DUE for a fixed time steps horizon can be decomposed into a series of static UE problems and solved sequentially. Thus, the DUE problem can be solved as a sequence of convex optimization problems.

INTRODUCTION

Dynamic traffic assignment (DTA) models have been studied since the seminal works of Merchant and Nemhauser in 1978 (*1, 2*). The principle of the *user equilibrium* (UE) (or Wardrop equilibrium) allocation, in which all travelers with the same origin-destination pair see the same travel time, was introduced by Wardrop (*3*) in the context of static traffic assignment and has been expanded to dynamic models by Beckman (*4*). The *system optimal* (SO), also introduced by Wardrop, corresponds to the minimization of the total travel time of all agents. Both the UE and the SO have many applications in traffic planning and *intelligent transportation systems* (ITS). The UE is used to represent the behavior of selfish agents and the SO is an upper bound in terms of network efficiency.

The static user equilibrium has been studied extensively in game theory (*5*), since it is a particular case of the Nash equilibrium concept. Several algorithms give arbitrarily good approximations of the static UE. Blum (*6*) presents a class of no-regret algorithms that give a strategy that, if applied by all agents, converges to a Nash equilibrium in static games, when latency functions are increasing, continuous and have bounded slopes. Fischer (*7*) presents another algorithm based on a replication-exploration protocol.

The *dynamic user equilibrium* (DUE) has been formulated using different hypothesis for the decision variables (route and/or departure time) and different models for the dynamics. Huang and Lam (*8*) study the simultaneous route and departure time choice problem using vertical queues to model the traffic. Lo and Stezo (*9*) formulate the DUE as a finite dimensional variational inequality problem and propose an alternating direction method to solve it. They also extend their model to handle elastic travel demand (*10*). Friesz and al. (*11*) present a continuous-time network loading procedure based on the *Lighthill-Williams-Richards* (LWR) model, formulate the DUE as a variational inequality problem and solve it using a fixed point algorithm. All of these methods are computationally complex even in simple networks.

We consider a macroscopic model (i.e. traffic flows and numbers of agents have continuous values. It is based on the assumption that one agent represents a negligible fraction of the overall traffic) based on the LWR partial differential equation (*12, 13*). Specifically, we use a Godunov discretization (*14*) of this equation, also known as the *Cell Transmission Model* (CTM) (*15, 16*) in transportation literature. We assume that the relationship between flow and density can be approximated to a first order using the trapezoidal fundamental diagram as seen in empirical studies (*17*).

We focus on the single source, single destination DUE problem with parallel paths, where the destination is not capacity restricted. As depicted in Fig. 1, each path is composed of an initial buffer of infinite capacity linked to a road network. Each path has its own buffer and the travel times of paths are independent from each other.

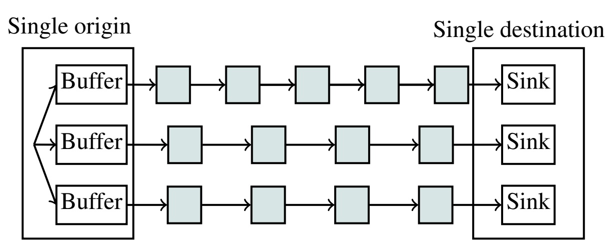

FIGURE 1 : An example of a parallel network with a single source and destination.

The contribution of the article is to show that the DUE problem can be solved as a sequence of static UE problems under the trapezoidal fundamental diagram hypothesis and a particular condition on the Godunov discretization of the road network. This reduces solving the DUE problem to a sequence of convex problems. A computationally efficient algorithm using any black box static UE allocation solver is given.

We first present the model of the traffic dynamics in section 3. Then, we prove that the UE assignment at a given time step does not depend on future demand, and thus describe a greedy algorithm to solve the DUE problem. Finally, section 5 gives implementation details and shows how the algorithm works on a simple three road parallel network.

PRELIMINARIES

Notations

Constants

Δt, T	Time discretization and number of time steps
$\pi(i)$	The predecessor of cell $i \in \mathscr{A} \setminus \mathscr{B}$
$\pi^{-1}(i)$	The successor of cell $i \in \mathscr{A} \setminus \mathscr{S}$
$D(k)$	Demand rate at the source for time step k
$F_i^{\max}$	Maximum inflow and outflow of cell i
L_i	Length of cell i
v_i	Free flow speed of cell i
w_i	Congestion wave speed of cell i
ρ_i^{jam}	Jam density of cell i
ρ_i^0	Initial density of cell i

Sets

$\mathscr{A}$	Set of cells (including buffers and sinks)
$\mathscr{A}_p$	Set of cells in path p
$\mathscr{B}$	Set of buffer cells
$\mathscr{S}$	Set of sink cells
$\mathscr{P}$	Set of parallel paths

Variables

$f_i^{\text{in}}(k)$	Total inflow of cell i at time step k
$f_i^{\text{out}}(k)$	Total outflow of cell i at time step k
$\rho_i(k)$	Density in cell i at time step k
$\sigma_i(k)$	Supply of cell i at time step k
$\delta_i(k)$	Demand of cell i at time step k
$\gamma_p(k)$	The split ratio for path p at time step k

For notational convenience, both sinks and buffers are considered as cells of unit length and infinite capacity. Thus, the density in a sink or in a buffer is equal to the number of agents in the cell. We also use $[\![0, T-1]\!] = \{0, \ldots, T-1\}$.

Dynamics model

The dynamics of the system govern the evolution of traffic over time and space. Each path has its own sink to be able to discriminate the number of agents that have reached the destination using that path. This article only considers the DUE problem for a parallel network. See (*18*) for a discussion of the general network problem.

Assumption 3.1 (First-in first-out (FIFO) property) *We assume that no agents leaving the origin at a time step $t > t'$ will overtake the agents that have left the origin at time step t'.*

Assumption 3.2 *The flux function defining the relationship between density and flow is given by the trapezoidal fundamental diagram shown in Fig. 2. This is a first order approximation of the empirical relationship between flow and density (17).*

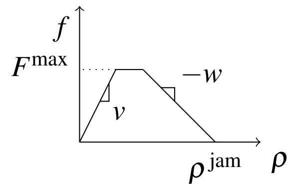

FIGURE 2 : Trapezoidal fundamental diagram.

Definition 3.1 (Supply and demand) *The supply of a cell i at time step k, denoted $\sigma_i(k)$, is the flow it can accept from its predecessor cell, while the demand $\delta_i(k)$ is the flow that is trying to leave the cell. Following the trapezoidal fundamental diagram and considering that both sinks and buffers are considered as cells of unit length, they are defined as:*

$$\sigma_i(k) = \min\left(F_i^{max}, v_i\rho_i(k)\right), \ \forall i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S}) \tag{1}$$

$$\sigma_i(k) = F_i^{max}, \ \forall i \in \mathscr{S} \tag{2}$$

$$\delta_i(k) = \min\left(F_i^{max}, w_i(\rho_i^{jam} - \rho_i(k))\right), \quad \forall i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S}) \tag{3}$$

$$\delta_i(k) = \frac{\rho_i(k)}{\Delta t}, \ \forall i \in \mathscr{B} \tag{4}$$

Note that buffers have no supply and the sinks have no demand.

Definition 3.2 (Split ratio) *The split ratio $\gamma_p(k)$ for path p at time step k is the fraction of the demand rate $D(k)$ that is taking path p at time step k.*

Definition 3.3 (Initial conditions) *The initial conditions for time step $k = 0$ are defined as:*

$$\rho_i(0) = \rho_i^0, \qquad \forall i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S}) \tag{5}$$

$$\rho_i(0) = D(0)\gamma_i(0)\Delta t, \qquad \forall i \in \mathscr{B} \tag{6}$$

$$\rho_i(0) = 0, \qquad \forall i \in \mathscr{S} \tag{7}$$

Definition 3.4 (Inflow and outflow) *Having all the densities at time step k, we compute the flows at time step k with:*

$$f_i^{out}(k) = f_{\pi^{-1}(i)}^{in}(k) = \min\left(\delta_i(k), \sigma_{\pi^{-1}(i)}(k)\right), \quad \forall i \in \mathscr{A} \setminus \mathscr{S} \tag{8}$$

Note that buffers have no inflow and sinks have no outflow.

Definition 3.5 (Forward system) *The state of the network at time step k is defined by all the densities of the cells. Having the state at time step k, the state at time step $k+1$ is computed using the following relations:*

$$\rho_i(k+1) = \rho_i(k) + \frac{\Delta t}{L_i}\left(f_i^{in}(k) - f_i^{out}(k)\right), \quad \forall i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S}) \tag{9}$$

$$\rho_i(k+1) = \frac{\Delta t}{L_i}\left(D(k+1)\gamma_i(k+1) - f_i^{out}(k)\right) + \rho_i(k), \qquad \forall i \in \mathscr{B} \tag{10}$$

$$\rho_i(k+1) = \rho_i(k) + \frac{\Delta t}{L_i} f_i^{in}(k), \ \forall i \in \mathscr{S} \tag{11}$$

To ensure the convergence of the solution of the discretized model to the solution of the continuous LWR equation when Δt converges to zero, the network must satisfy the *Courant–Friedrichs–Lewy* (CFL) conditions, which are standard requirements in numerical analysis (*14, 19*).

Requirement 3.1 (CFL conditions)

$$v_i \leq \frac{L_i}{\Delta t}, \qquad \forall i \in \mathscr{A} \setminus \mathscr{S} \qquad \text{(CFL 1)}$$

$$w_i \leq \frac{L_i}{\Delta t}, \qquad \forall i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S}) \qquad \text{(CFL 2)}$$

We also introduce an additional requirement that guarantees that the space discretization does not result in a degenerate dynamics model where some agents might never leave the network.

Requirement 3.2 (Non-exponential decrease condition)

$$v_i = \frac{L_i}{\Delta t}, \ \forall i \in \mathscr{A} \setminus \mathscr{S}$$

Interpretation: If $v_i < \frac{L_i}{\Delta t}$, we can have an exponential decrease of the density in cell i when it should be emptied in a finite number of time steps. Indeed, taking the case of a cell in free-flow (the demand is limiting the outflow) without inflow, we have $\rho_i(k+1) = \rho_i(k) - \frac{\Delta t}{L_i}\rho_i(k)v_i$ which gives $\rho_i(k+t) = \rho_i(k)(1 - \frac{v_i \Delta t}{L_i})^t$. Thus, cell i never empties, and this is a model limitation. Requirement 3.2 prevents this degenerate case.

Remark 3.1 *The requirement $v_i = \frac{L_i}{\Delta t}$ for all cells i adds some rigidity in the framework proposed here. Indeed, for a given road segment, one has to divide it into cells of exact length $v_i \Delta t$, but in most cases a final cell of length $l \in]v_i \Delta t, 2 v_i \Delta t[$ will remain. One can accept to round the length of the road to a multiple of $v_i \Delta t$ or focus on fixing the behavior of the last cell by modifying the dynamics to ensure that if the cell can be emptied in only $\lceil \frac{L_i}{v_i} \rceil$ steps, it does so. See remark 5.1 for more details.*

Definition of the travel time

Definition 3.6 (Demand rate) *The demand rate $D(k)$ at time step k is the rate of agents leaving the origin between time $k\Delta t$ and $(k+1)\Delta t$. The demand rate at time step k on path p is $D_p(k) = \gamma_p(k) D(k)$.*

Definition 3.7 (Cumulative departure curve (CDC)) *The cumulative departure curve (CDC) at time step k for path p is the count of agents, having left the origin at some time step preceding time step k (included). It is defined as:*

$$CDC_p(k) = \Delta t \sum_{t=0}^{k} D_p(t) \qquad (12)$$

If no path is specified, it refers to the total cumulative count on all paths $CDC(k) = \sum_{p \in \mathscr{P}} CDC_p(k)$. Let $CDC_p(-1) = 0$ for notational convenience.

Definition 3.8 (Cumulative arrival curve (CAC)) *The cumulative arrival curve (CAC) at time step k on path p is the total count of agents that have arrived at the sink of path p by time k and have left the origin at a time step $t \geq 0$. We use the notation $f_i^{out}(k)$ for the outflow of cell i at time step k. Then, with s being the sink at the end of path p and considering the fact that buffers are cells of unit length, the CAC is defined as:*

$$\begin{aligned} CAC_p(k) &= -\sum_{i \in \mathscr{A}_p} \rho_i(0) + \sum_{t=0}^{k-1} f_{\pi(s)}^{out}(t)\,\Delta t \\ &= -\sum_{i \in \mathscr{A}_p} \rho_i(0) + \rho_s(k) \end{aligned} \qquad (13)$$

The notational simplification of the double summation into $\rho_s(k)$ is obtained using the fact that sinks are modeled as cells of unit length (see equation (11)). If no path is specified, it refers to the total cumulative count on all paths $CAC(k) = \sum_{p \in \mathscr{P}} CAC_p(k)$.

This method of using cumulative arrival and departure curves to determine the average travel time is also discussed by Han (*20*).

Definition 3.9 (Aggregate and average travel time) *The aggregate travel time $ATT_{p,k}$ of the agents entering path p at time step k is the shaded area depicted in Fig. 3. This area is delimited by the $x = k$ vertical line, the $y = CDC_p(k)$ and $y = CDC_p(k-1)$ horizontal lines, and the $CAC_p(\cdot)$ curve. Denoting $t_{first}(p,t) = \min\{t \mid CAC_p(t) > CDC_p(k-1)\}$ and $t_{last}(p,t) = \min\{t \mid CAC_p(t) \geq CDC_p(k)\}$ we have:*

$$\begin{aligned} ATT_{p,k} &= (CDC_p(k) - CDC_p(k-1))\,(t_{last}(p,t) - k) \\ &\quad - \sum_{t=t_{first}(p,t)}^{t_{last}(p,t)-1} (CAC_p(t) - CDC_p(k-1)) \end{aligned} \tag{14}$$

The average travel time $TT_{p,k}$ of agents leaving the origin at time step k and taking path p is computed as the aggregate travel time of these agents divided by the total number of agents $\gamma_p(k)D(k)\Delta t$ entering path p at time step k. Both $ATT_{p,k}$ and $TT_{p,k}$ are functions of the initial densities and split ratios, but for notational convenience, only the dependence with respect to $D_p(k)$ will be explicit (i.e. we use $ATT_{p,k}(D_p(k))$ and $TT_{p,k}(D_p(k))$). If $D_p(k) = 0$ we define the travel time $TT_{p,k}(0) = \lim_{\eta \to 0^+} TT_{p,k}(\eta)$.

Note that this average travel time is the average number of time steps it takes to reach the sink. To get the actual time, one has to multiply the here above defined average travel time by Δt.

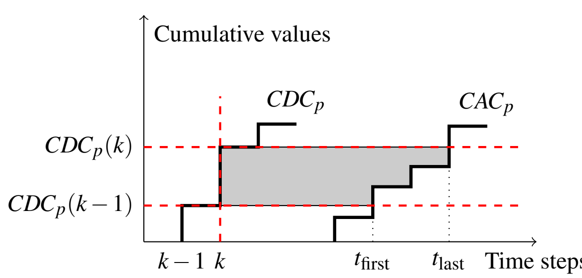

FIGURE 3 : The shaded area is the aggregate travel time of the agents leaving at time step k and taking path p.

Problem definition

The allocation problem can be solved under different information models. Two such models are:

1. Only the current network state and demand at the current time step are known by the agents. In this approach, referred to as reactive assignment, agents leaving at time step k, knowing only the state of the network at that time step, take the path which optimizes their instantaneous travel time. Their instantaneous travel time may differ from the effective travel time that they will experience (the effective travel time is defined as the travel time experienced by the agents upon arriving.).

2. The agents have perfect knowledge of the variables of the problem (present and future) and will determine the user equilibrium split ratios using that perfect knowledge. In this case, referred

to as predictive assignment, agents leaving at time step k have absolute knowledge (in particular they know the present and future characteristics of the network and the present and future demand) and their predicted travel time is what they will experience. Thus, in a UE, agents will allocate themselves among the paths in a manner that they will have the same effective travel time.

We will prove in Section 4.1 that both approaches give the same result in the case of independent parallel paths, due to the fact that future flow does not affect the travel time of agents already in the network.

Definition 3.10 (User Equilibrium) *In a user equilibrium (3), no agent has an incentive to change his or her strategy (i.e. path) unilaterally. It is equivalent to a Nash equilibrium in nonatomic selfish routing, see Definition 18.1 in (21). In a macroscopic model, in which one agent represents a negligible fraction of the overall traffic, an agent changing its path does not modify any path travel time. Thus, a UE is an allocation in which at each time step k, all the paths with non zero flow ($\gamma_p(k) > 0$) must have the same travel time to the destination, and the unused paths ($\gamma_p(k) = 0$) must have a travel time higher than or equal to the travel time of the used paths:*

$$\forall k \in [\![0, T-1]\!], \exists v_k, \forall p \in \mathscr{P}, \begin{cases} \gamma_p(k) > 0 \Rightarrow TT_{p,k} = v_k \\ \gamma_p(k) = 0 \Rightarrow TT_{p,k} \geq v_k \end{cases} \tag{15}$$

Definition 3.11 (Physical split ratio) *A set of physical split ratios satisfies:*

$$\gamma_p(k) \geq 0 \qquad \forall k \in [\![0, T-1]\!],\ \forall p \in \mathscr{P} \tag{16}$$

$$\sum_{p \in \mathscr{P}} \gamma_p(k) = 1 \qquad \forall k \in [\![0, T-1]\!] \tag{17}$$

PROBLEM STATEMENT: Compute a set of physical split ratios $(\gamma_p(k))_{p \in \mathscr{P},\ k \in [\![0, T-1]\!]}$ that result in a user equilibrium allocation of the agents.

SPLIT RATIO OPTIMIZATION

Time decoupling of the UE split ratio optimization

Theorem 4.1 *Under assumption 3.2 and requirement 3.2, the travel time of the agents leaving the origin at some time step k does not depend on the agents leaving at future time steps. While this fact might seem intuitive in the case of traffic, a proof needs to be made that it is indeed true given the mathematical model (1) to (11).*

We first introduce some notations. For a given time step $k \in [\![0, T-1]\!]$ and a path $p \in \mathscr{P}$, let $\mathscr{E}_{p,k} \subset (\mathbb{R}_+)^T$ defined as:

$$\mathscr{E}_{p,k} = \left\{ \left(D'_p(t')\right)_{t' \in [\![0, T-1]\!]} \mid D'_p(t) = D_p(t),\ \forall t \leq k \right\} \tag{18}$$

be the set of demand profiles for path p that contain the current demand history ($t \leq k$) and all possible future demands ($k < t \leq T-1$). Let $D_k^{\mathrm{NF}} \in \mathscr{E}_{p,k}$ (No Future demand) be defined as:

$$D_{p,k}^{\mathrm{NF}}(t) = \begin{cases} D_p(t) & \text{if } t \leq k \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

For any d $\in \mathscr{E}_{p,k}$, we denote by $\rho_i^{\mathrm{d}}(k)$ (respectively $\sigma_i^{\mathrm{d}}(k)$, $f_i^{\text{out, d}}(k)$ and $f_i^{\text{in, d}}(k)$) the density (respectively supply, outflow and inflow) of cell i at time step k in the state profile (i.e. values of the densities and flows for all time steps and all cells due to the dynamics of the system) due to the demand profile d. The individual departure times of all the agents in cell i at time step t can be determined explicitly due to the FIFO property. Thus, let $\rho_{i,t'}^{\mathrm{d}}(t)$ be the contribution of the vehicles left at time step t' to the total density $\rho_i^d(t)$ of cell i at time step t in the state profile using the sequence of demand d.

For any d $\in \mathscr{E}_{p,k'}$ and for any cell $i \in \mathscr{A}_p \setminus \mathscr{S}$, we define

$$t_{\min}^{\mathrm{d}}(k',i) = \min\{t \mid \rho_{i,k'}^{\mathrm{d}}(t) \neq 0\} \tag{20}$$

$$t_{\max}^{\mathrm{d}}(k',i) = \max\{t \mid \rho_{i,k'}^{\mathrm{d}}(t) \neq 0\} \tag{21}$$

The value $t_{\min}^{\mathrm{d}}(k',i)$ (respectively $t_{\max}^{\mathrm{d}}(k',i)$) can be interpreted as the first (respectively last) time step when a portion of the vehicles left at time step k' is in cell i in the state profile due to the demand profile d.

Proof of Theorem 4.1: Let $k \in [\![0, T-1]\!]$ be a time step and $p \in \mathscr{P}$ be a path. We want to prove that the future demands on path p (i.e. $D_p(t),\ \forall t > k$) do not modify the travel time of the agents leaving the origin at time step k.

Let d $\in \mathscr{E}_{p,k}$ be a fixed demand profile for path p. For notational convenience, we will refer to $D_{p,k}^{\mathrm{NF}}$ as NF in the exponents (e.g $\rho_i^{D_{p,k}^{\mathrm{NF}}}(t) = \rho_i^{\mathrm{NF}}(t)$).

We prove the following properties using induction over the cells from the buffer to the last non-sink cell of path p:

$$t_{\min}^{\mathrm{NF}}(k,i) = t_{\min}^{\mathrm{d}}(k,i) \tag{22}$$

$$t_{\max}^{\mathrm{NF}}(k,i) = t_{\max}^{\mathrm{d}}(k,i) \tag{23}$$

$$f_i^{\text{out, NF}}(t) = f_i^{\text{out, d}}(t),\ \forall t < t_{\max}^{\mathrm{NF}}(k,i) \tag{24}$$

$$f_i^{\text{out, NF}}\left(t_{\max}^{\mathrm{NF}}(k,i)\right) \leq f_i^{\text{out, d}}\left(t_{\max}^{\mathrm{NF}}(k,i)\right) \tag{25}$$

a) Initialization (i = buffer)

• *Proof of (22)*: Cell i is now the buffer for path p. Since both the sequences of the demand share the same past history until time step k (i.e. $D_{p,k}^{\mathrm{NF}}(t) = \mathrm{d}(t),\ \forall t \leq k$) we have:

$$t_{\min}^{\mathrm{NF}}(k,i) = t_{\min}^{\mathrm{d}}(k,i) = k \tag{26}$$

$$f_i^{\text{out, NF}}(t) = f_i^{\text{out, d}}(t), \qquad \forall t < t_{\min}^{\mathrm{NF}}(k,i) = k \tag{27}$$

$$\sigma_{\pi^{-1}(i)}^{\mathrm{NF}}(t) = \sigma_{\pi^{-1}(i)}^{\mathrm{d}}(t), \qquad \forall t < t_{\min}^{\mathrm{NF}}(k,i) = k \tag{28}$$

• *Proof of (24)*: In the state profile with respect to $D_{p,k}^{\mathrm{NF}}$, if the agents that entered the network at time step k pass through cell i in at least more than one time step (i.e. $t_{\max}^{\mathrm{NF}}(k,i) \neq t_{\min}^{\mathrm{NF}}(k,i)$) we have for $t \in [\![t_{\min}^{\mathrm{NF}}(k,i), t_{\max}^{\mathrm{NF}}(k,i) - 1]\!]$ that the outflow is supply or capacity limited (i.e. $\sigma_{\pi^{-1}(i)}^{\mathrm{NF}}(t) < \delta_i(t)$ or $F_i^{\max} < \delta_i(t)$):

$$f_i^{\text{out, NF}}(t) = \min\left(F_i^{\max}, \sigma_{\pi^{-1}(i)}^{\mathrm{NF}}(t)\right) \tag{29}$$

Indeed, if this were not the case, there would exist $t \in [\![t_{\min}^{\mathrm{NF}}(k,i), t_{\max}^{\mathrm{NF}}(k,i) - 1]\!]$ such that $f_i^{\text{out, NF}}(t) = \delta_i(t) < \min\left(F_i^{\max}, \sigma_{\pi^{-1}(i)}^{\mathrm{NF}}(t)\right)$. However, because of Requirement 3.2, this means that the cell will be emptied in one time step and then we have $t = t_{\max}^{\mathrm{NF}}(k,i)$, which is a contradiction. Note that requirement 3.2 is essential for this theorem to hold.

Equation (28) gives that the supply in the two state profiles up to (and excluding) $t^{\mathrm{NF}}_{\min}(k,i)$ are equal. Due to equation (29), the ouflow only depends on the downstream cell and the outflow will be the same until $t^{\mathrm{NF}}_{\max}(k,i)-1$ (i.e. equation (30)) which also proves (31).

$$f_i^{\text{out, NF}}(t) = f_i^{\text{out, d}}(t),\ \forall t < t^{\mathrm{NF}}_{\max}(k,i) \tag{30}$$

$$t^{\mathrm{NF}}_{\max}(k,i) \le t^{\mathrm{d}}_{\max}(k,i) \tag{31}$$

• *Proof of (23) and (25)*: At time step $t^{\mathrm{NF}}_{\max}(k,i)$, in the state profile using the demand profile d, since the outflow is an increasing function of the density, we have the following implication:

$$\begin{aligned}\rho_i^{\mathrm{NF}}\left(t^{\mathrm{NF}}_{\max}(k,i)\right) \le \rho_i^{\mathrm{d}}\left(t^{\mathrm{NF}}_{\max}(k,i)\right) \Rightarrow \\ f_i^{\text{out, NF}}\left(t^{\mathrm{NF}}_{\max}(k,i)\right) \le f_i^{\text{out, d}}\left(t^{\mathrm{NF}}_{\max}(k,i)\right)\end{aligned} \tag{32}$$

The first assumption is true because $D^{\mathrm{NF}}_{p,k}(t) = 0$ for $t > k$. Due to the FIFO property, it follows that $t^{\mathrm{NF}}_{\max}(k,i) = t^{\mathrm{d}}_{\max}(k,i)$.

b) Induction: Let $i \in \mathscr{A} \setminus (\mathscr{B} \cup \mathscr{S})$. We assume the properties (22) to (25) for cell $\pi(i)$.

• *Proof of (22)*: Let $t_{\pi(i)} = t^{\mathrm{NF}}_{\min}(k,\pi(i))$ for notational convenience. Due to the FIFO property, the number of agents at time step $t_{\pi(i)}$ in cell $\pi(i)$ who are ahead of the agents that left the origin at time step k is $m = L_{\pi(i)} \sum_{t'<k} \rho^{\mathrm{NF}}_{\pi(i),t'}\left(t_{\pi(i)}\right)$. This value is the same for the state profile using d because of (24) for cell $\pi(i)$. Then because of equations (24) and (25) for cell $\pi(i)$ we have:

$$\begin{aligned} t^{\mathrm{NF}}_{\min}(k,i) &= t^{\mathrm{d}}_{\min}(k,i) \\ &= 1 + \min\left\{ t \ge t_{\pi(i)} \mid \sum_{t'=t_{\pi(i)}}^{t} f^{\text{out, NF}}_{\pi(i)}(t') > \frac{m}{\Delta t} \right\} \end{aligned} \tag{33}$$

Since at time step $t^{\mathrm{NF}}_{\max}(k,\pi(i))$, there are still agents that left the origin at time step k in $\pi(i)$, for any $t > k$ we have $\rho^{\mathrm{d}}_{i,t}(t') = 0,\ \forall t' \le t^{\mathrm{NF}}_{\max}(k,\pi(i))$. Thus, we have the following equations that are analogous to (27) and (28):

$$f_i^{\text{out, NF}}(t) = f_i^{\text{out, d}}(t), \qquad \forall t < t^{\mathrm{NF}}_{\max}(k,\pi(i)) + 1 \tag{34}$$

$$\sigma^{\mathrm{NF}}_{\pi^{-1}(i)}(t) = \sigma^{d}_{\pi^{-1}(i)}(t), \qquad \forall t < t^{\mathrm{NF}}_{\max}(k,\pi(i)) + 1 \tag{35}$$

• *Proof of (23), (24), (25):* The rest of the proof follows directly from the proof for the base case (i = *buffer*), by replacing $t^{\mathrm{NF}}_{\min}(k,i)$ with $t^{\mathrm{NF}}_{\max}(k,\pi(i)) + 1$.

c) Application of the result to the last cell prior to the sink Let s be the sink of path p. Let $t_{\max} = t^{\mathrm{NF}}_{\max}(k,\pi(s))$ for notational convenience. Applying the inductive proof for all cells except for the sink, we have:

$$f^{\text{out, NF}}_{\pi(s)}(t) = f^{\text{out, d}}_{\pi(s)}(t),\ \forall t < t^{\mathrm{NF}}_{\max}(k,\pi(s)) \tag{36}$$

$$f^{\text{out, NF}}_{\pi(s)}(t_{\max}) \le f^{\text{out, d}}_{\pi(s)}(t_{\max}) \tag{37}$$

Thus, the travel time of the agents leaving the origin at time step k, computed using (12) and (13), are equal in the state profiles due to d or NF. The geometrical interpretation is that any modification of the demand profile after time step k does not modify the CDC_p and CAC_p curves below the line $y = CDC_p(k)$.

Theorem 4.2 (Decoupling of the UE optimization) *The DUE for* $k \in [\![0, T-1]\!]$ *can be solved as* T *sequential static UE problems.*

Proof (by induction): Let $t \in [\![0, T-1]\!]$ be an arbitrary time step. We suppose that the split ratios for the time steps $k \in [\![0, t-1]\!]$ correspond to a user equilibrium:

$$\forall k \in [\![0, t-1]\!], \exists v_k, \forall p \in \mathscr{P}, \begin{cases} \gamma_p(k) > 0 \Rightarrow TT_{p,k} = v_k \\ \gamma_p(k) = 0 \Rightarrow TT_{p,k} \geq v_k \end{cases} \quad (38)$$

Given the split ratios for time steps $k \in [\![0, t-1]\!]$, we want to find some user equilibrium split ratios for time steps $k \in [\![0, t]\!]$. There exists v_t such that:

$$\forall p \in \mathscr{P}, \begin{cases} \gamma_p(t) > 0 \Rightarrow TT_{p,t} = v_t \\ \gamma_p(t) = 0 \Rightarrow TT_{p,t} \geq v_t \end{cases} \quad (39)$$

Then, because of Theorem 4.1, as the future will not modify the already computed travel times, (38) will still hold. Equations (38) and (39) prove that we have a UE for time steps $k \in [\![0, t]\!]$.

Thus, we can obtain a user equilibrium allocation for time steps $k \in [\![0, T-1]\!]$ by solving T static user equilibrium problems.

Theorem 4.3 (Uniqueness of the average travel time) *There can be several allocations resulting in a user equilibrium. However, under assumption 3.2 and requirement 3.2 there is only one average travel time for all paths with non zero flow at each departure time, in the case of the single source, single destination parallel network.*

Proof: Because of Theorem 4.2, the optimization of one step is equivalent to the Nash equilibrium in a Nonatomic Selfish Routing game (*21*). Theorem 18.8 of that chapter proves that the cost of the roads for any static UE split ratios is the same.

Continuity of the travel time

Theorem 4.4 (Continuity of the average travel time) *For any time step k and path p, the average travel time* $TT_{p,k}(D(k)\gamma_p(k))$ *is continuous with respect to* $\gamma_p(k)$.

Proof: Equations (1) to (11) are continuous with respect to the variables of the problems (i.e. $f_i^{\text{in}}(t)$, $f_i^{\text{out}}(t)$, $\rho_i(t)$, $\sigma_i(t)$, $\delta_i(t)$ and $\gamma_{p'}(t)$ for all $i \in \mathscr{A}$, $t \in [\![0, T-1]\!]$ and $p' \in \mathscr{P}$). Let $\rho_{i,t'}(t)$ be the contribution of the vehicles left at time step t' to the total density $\rho_i(t)$ of cell i at time step t. Then, for $i \in \mathscr{A}_p \setminus \mathscr{S}$, we can explicitly give the value of $\rho_{i,t'}(t)$ because of the FIFO property:

$$\rho_{i,t'}(t) = \begin{cases} 0 \text{ if } \sum_{t''=0}^{t'-1} f_i^{\text{in}}(t'') \leq CDC_p(t-1) \\ 0 \text{ if } \sum_{t''=0}^{t'-1} f_i^{\text{out}}(t'') \geq CDC_p(t) \\ CDC_p(t) - \frac{\Delta t}{L_i} \sum_{t''=0}^{t'-1} f_i^{\text{out}}(t'') \text{ otherwise} \end{cases} \quad (40)$$

which proves that $\rho_{i,t'}(t)$ is continuous with respect to $\gamma_p(k)$. Note that, for the buffer b of path p, we note $f_b^{\text{in}}(t'') = \gamma_p(t'')D(t'')$.

The aggregate travel time computed from (14) is equivalent to (see Equation 2.5,page 28, in (*22*) for the proof):

$$ATT_{p,k} = \Delta t \sum_{t=0}^{T-1} \sum_{i \in \mathscr{A} \setminus \mathscr{S}} \rho_{i,k}(t) L_i \quad (41)$$

This proves that $ATT_{p,k}$ is continuous with respect to $\gamma_p(k)$. The average travel time is computed as $TT_{p,k} = \frac{ATT_{p,k}(\gamma_p(k))}{D_p(k)\Delta t}$, as long at the denominator is non zero.

• If $D(k) \neq 0$: we first prove the continuity for $\gamma_p(k) \in]0,1]$. The continuity of $TT_{p,k}$ at $D(k)\gamma_p(k)$ is trivial as the ratio of two continuous functions, in which the denominator converges to a non zero value.

We now prove continuity at $\gamma_p(k) = 0$. We use $D(k)\gamma_p(k) = \nu > 0$ for the number of agents. In that case, because of requirement 3.2, the network empties in a finite number of time steps. Among the agents leaving the origin at time step k and taking path p, some amount C experiences the smallest travel time $TT_{\min}(p,k) = \min\{t \mid CAC_p(t) > CDC_p(k-1)\} - k$. Theorem 4.1 proves that if we use $\gamma_p(k)$ such that $D(k)\gamma_p(k) \leq \frac{C}{D(k)\Delta t}$, all agents leaving at time step k and taking path p will still experience $TT_{\min}(p,k)$ (i.e. the area in Fig. 3 is a rectangle). This means that there exist some C' and m such that $ATT_{p,k}(\eta) = m\eta\Delta t,\ \forall \eta \in]0, C']$. This proves that $\lim_{\eta \to 0+} ATT_{p,k}(\eta) = \lim_{\gamma \to 0+} ATT_{p,k}(D_p(k)\gamma)$ exists.

• If $D(k) = 0$: then we have that $ATT_{p,k}(D_p(k)) = ATT_{p,k}(0), \forall \gamma_p(k)$ which proves that $ATT_{p,k}$ is continuous with respect to $\gamma_p(k)$.

Corollary 4.1 *In this context, the DUE is a sequence of convex problems.*

Proof: For a given time step k, the travel time functions $\left(TT_{p,k}\right)_p$ being continuous and increasing with respect to the split ratio $\gamma_p(k)$, the static problem is convex by (*23*).

IMPLEMENTATION

Computation of the travel time

The average travel time is computed using Algorithm 1. The function `computeDynamics(`p`, `k`)` computes both the flows at time step k and the densities at time step $(k+1)$, for a given path p, given the densities at time k (see equations (8) to (11)). The function `ComputeShadedArea(p, k)` computes the aggregate travel time represented in Fig. 3 by formula (14).

If no agents are allocated to path p at time step k, we approximate the average travel time by adding an artificial infinitesimal demand rate η.

Remark 5.1 *If requirement 3.2 does not hold, the exponential decrease of the density in some cells may prevent some agents from reaching the destination. To mitigate this phenomenon, we can add more demand at the time step without counting it in $CDC_p(k)$.*

Algorithm 1 Computes $TT_{p,k}(\gamma_p)$

Input: A time step k, a path p and split ratio $\gamma_p(k)$
Output: $TT_{p,k}(\gamma_p)$
  if $\gamma_p = 0$ **then**
    $\gamma_p = \eta$ % Used to avoid division by 0
  end if
  $CDC_p(k) = CDC_p(k-1) + D(k)\gamma_p(k)\Delta t$
  $t_{\text{last}} = k$
  while $CAC_p(t_{\text{last}}) < CDC_p(k)$ **do**
    `computeDynamics`(p, t_{last})
    $t_{\text{last}} = t_{\text{last}} + 1$
  end while
  $TT =$ `ComputeShadedArea`$(p,k) / (D_p(k)\Delta t)$
  return TT

Greedy algorithm

There are many algorithms for computing a static user equilibrium allocation, such as no-regret algorithms (*6*) and adaptive sampling methods (*7*).

In our implementation (an open source implementation is available at `https://github.com/calpath/DTA-Simulator`.), we opted for an exponentially weighted average forecaster (*24*) (a no-regret algorithm). In this case, an ε-UE is a solution such that $\sum_{p\in\mathscr{P}} \gamma_p(k) TT_{p,k} \leq \varepsilon + \min\left(TT_{p,k}\right)_{p\in\mathscr{P}}$. The function `isOptimalSplitRatio()` in Algorithm 2 returns true if this stopping criterion is satisfied. For this algorithm to converge, the travel time functions must be increasing, continuous and have bounded slopes. These conditions are verified in our model (see Theorem 4.4 for the continuity).

Algorithm 2 Optimizes split ratios at time step k to get an ε-UE

Input: Optimal values of the split ratios for the steps $s \in [\![0, k-1]\!]$
Output: Optimal split ratios for the steps $s \in [\![0, k]\!]$

Initialize arbitrarily $(\gamma_p(k))_{p\in\mathscr{P}}$ such that $\gamma_p(k) \neq 0$ for all $p \in \mathscr{P}$ and $\sum_{p\in\mathscr{P}} \gamma_p(k) = 1$
Define $\forall p \in \mathscr{P},\ \beta_p(1) = \gamma_p(k)$
$n = 1$
while not `isOptimalSplitRatio()` **do**
  $\beta_p(n+1) = \beta_p(n) \exp\left(-\varepsilon TT_{p,k}(\gamma_p(k))\right),\ \forall p \in \mathscr{P}$
  $\gamma_p(k) = \frac{\sum_{1\geq t\geq n+1} \beta_p(t)}{n+1},\ \forall p \in \mathscr{P}$
  $\gamma_p(k) = \frac{\gamma_p(k)}{\sum_{p'\in\mathscr{P}} \gamma_{p'}(k)},\ \forall p \in \mathscr{P}$ % Normalization
  $n = n+1$
end while

Example

The algorithm is tested on the simple three paths parallel network depicted in Fig. 1. The time horizon is $T = 11$ with a discretization of $\Delta t = 1$, and the cells share the same characteristics: $L = 1$, $w = 0.4$, $\rho^{\text{jam}} = 8$ and $F^{\max} = 2$. The details are given in the tables 1, 2 and 3. The maximum flow of a path is the maximum inflow of the sink at the end of the road. Fig. 4 gives the results from solving the DUE with $\varepsilon = 10^{-2}$.

TABLE 1 : Description of the roads.

Path	Number of cells	Maximum inflow of the sink
1	5	2
2	4	1.5
3	4	1

TABLE 2 : Description of the demand.

Time step	0	1	2	3	4	5	6	7	8	9	10
$D(k)$	1	2	3	4	5	6	5	4	3	2	1

TABLE 3 : Number of iterations necessary to get convergence.

Time step	0	1	2	3	4	5	6	7	8	9	10
Iterations	6	6	7	8	8	5	6	6	5	2	6

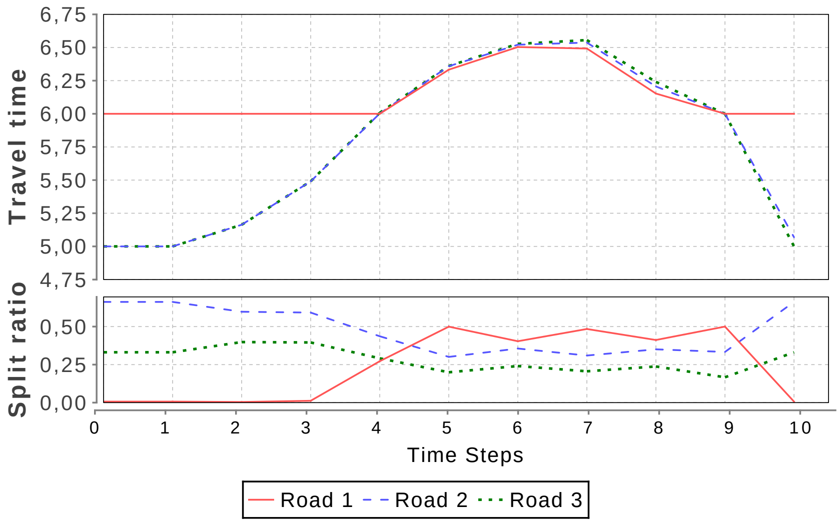

FIGURE 4 : Optimized travel times and split ratios.

CONCLUSION

We have showed that the dynamic user equilibrium problem for parallel independent networks can be solved as a sequence of static UE problems, when the dynamics are given by a Godunov discretization of the Lighthill-Williams-Richards partial differential equation with a trapezoidal fundamental diagram. An increasing demand function at the cell level and the non-exponential decrease condition are also required to decouple the dynamic UE into static problems. This decomposition proof extends to any fundamental diagram with an increasing demand function. The parallel network requirement can also be relaxed to more general networks where future demands do not interact with the travel time of the agents already in the network, but the types of networks that satisfy this requirement are still very limited.

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